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Continuity of solutions of the translation equation

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Summary. Assuming that X is a metric space and $F\colon (0,\infty)\times X\to X$ satisfies the translation equation

$$F(s+t,x)=F(t,F(s,x))$$

for $s,t\in(0,\infty)$ and $x\in X$, we show that:

1. If F is separately continuous, then it is continuous.
2. If X is separable and F is Carathéodory, then F is continuous.

We also show that if X is merely a topological, possibly separable, space, then a separately continuous mapping $F\colon (0,\infty)\times X\to X$ satisfying the above translation equation may fail to be continuous.

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The aim of this note is to improve results of [1], cf. also [2; 22.4], on continuity of Carathéodory solutions of the translation equation

$$F(s+t,x)=F(t,F(s,x)) \tag{1}$$

and also to study separately continuous solutions of (1). Here the term "Carathéodory" refers to a function of two variables which is measurable in the first variable and continuous in the second one, with measurability understood either in the sense of Lebesgue or that of Baire. In what follows, the only measurable functions — treated as functions of one variable — will be those mapping a set of reals into a metric space. Recall that any such function is Lebesgue (Baire) measurable if the inverse image of every open set via this function is Lebesgue measurable (has the Baire property, respectively). Most of our results will concern solutions of the Pexider analogue of (1), i.e., functions satisfying

$$h(s+t,x)=g(t,f(s,x)) \qquad \text{for} \quad s,t\in(0,\infty) \quad \text{and} \quad x\in X. \tag{2}$$

Our development relies on [1; Theorem] and the main results read as follows.

Theorem 1. *Assume X and Z are metric spaces and Y is a separable metric space. Let $f\colon (0,\infty)\times X \to Y$, $g\colon (0,\infty)\times Y \to Z$ and $h\colon (0,\infty)\times X \to Z$ be continuous as functions of the second variable and satisfy* (2)*. Fix $t_0 \in (0,\infty)$.*

(A) *If there exists a Lebesgue measurable set $M \subset (0,t_0)$ with positive Lebesgue measure such that $f(\cdot,x)|_M$ is Lebesgue measurable for every $x \in X$, then $h|_{(t_0,\infty)\times X}$ is continuous.*

(B) *If there exists a set $M \subset (0,t_0)$ of second category with the Baire property such that $f(\cdot,x)|_M$ is Baire measurable for every $x \in X$, then $h|_{(t_0,\infty)\times X}$ is continuous.*

Theorem 2. *Assume X, Y and Z are metric spaces. Let $f\colon (0,\infty)\times X \to Y$, $g\colon (0,\infty)\times Y \to Z$ and $h\colon (0,\infty)\times X \to Z$ be continuous as functions of the second variable and satisfy* (2)*. If f is continuous also in the first variable, then h is continuous.*

Proof of Theorem 1. If X is compact, then the result is a particular case of [1; Theorem]. To treat the general case, it suffices to show (noting that for each convergent sequence in X the set comprising all elements of the sequence and the limit of the sequence is compact) that $h|_{(t_0,\infty)\times K}$ is continuous for any compact set $K \subset X$. But this is straightforward: given a compact set $K \subset X$, the continuity of $h|_{(t_0,\infty)\times K}$ follows immediately upon applying the result in the compact case to $f|_{(0,\infty)\times K}$, g, and $h|_{(0,\infty)\times K}$.

Proof of Theorem 2. It suffices to prove that $h|_{(0,\infty)\times K}$ is continuous for any countable compact set $K \subset X$. Given a countable compact set $K \subset X$, define a subspace Y_0 of Y by

$$Y_0 = f((0,\infty)\times K).$$

Being a countable union of separable subspaces, namely the union of all the $f((0,\infty)\times\{x\})$ with $x \in K$, Y_0 is separable. Applying Theorem 1 to $f|_{(0,\infty)\times K}$, $g|_{(0,\infty)\times Y_0}$ and $h|_{(0,\infty)\times K}$, we infer that $h|_{(0,\infty)\times K}$ is continuous.

As immediate consequences we obtain the following corollaries.

Corollary 1. *Assume X is a separable metric space and $F\colon (0,\infty)\times X \to X$ is a solution of* (1) *that is continuous in the second variable. Let $t_0 \in (0,\infty)$.*

(A) *If there exists a Lebesgue measurable set $M \subset (0,t_0)$ with positive Lebesgue measure such that $F(\cdot,x)|_M$ is Lebesgue measurable for every $x \in X$, then $F|_{(t_0,\infty)\times X}$ is continuous.*

(B) *If there exists a set $M \subset (0,t_0)$ of second category with the Baire property such that $F(\cdot,x)|_M$ is Baire measurable for every $x \in X$, then $F|_{(t_0,\infty)\times X}$ is continuous.*

Corollary 2. *If X is a metric space, then any solution $F\colon (0,\infty)\times X \to X$ of* (1) *that is continuous in each variable is continuous.*

The following example shows that in Corollary 2 the condition for X to be a metric space cannot be replaced by the requirement that X be a topological, possibly even separable, space.

Example. Let X be the space of all real continuous functions on the space $\mathbb{R}$ of reals, equipped with the Tychonoff topology (the topology of pointwise convergence). Note that X is separable — the set of all polynomials with rational coefficients is a countable dense subset of X. Define the mapping $F\colon \mathbb{R} \times X \to X$ by

$$F(s,x)(\omega) = x(\omega + s) \qquad (\omega \in \mathbb{R}).$$

Clearly, F satisfies (1). We will show that it is separately continuous and discontinuous at each point of $\mathbb{R} \times X$.

For $s \in \mathbb{R}$ let $\pi_s\colon X \to \mathbb{R}$ denote the projection:

$$\pi_s(x) = x(s) \qquad (x \in X).$$

Then

$$\pi_s \circ F(t,\cdot) = \pi_{s+t} \qquad \text{for} \quad s,t \in \mathbb{R},$$

which proves the continuity of $F(t,\cdot)$ for any $t \in \mathbb{R}$. If $x \in X$, then

$$(\pi_s \circ F(\cdot,x))(\omega) = x(\omega + s) \qquad \text{for} \quad s,\omega \in \mathbb{R}$$

and so $F(\cdot,x)$ is continuous.

Fix $(s_0,x_0) \in \mathbb{R} \times X$ and put

$$W = \{x \in X :\ |x(0) - F(s_0,x_0)(0)| < 1\}.$$

Clearly, W is a neighbourhood of $F(s_0,x_0)$. We will show that $F(U) \not\subset W$ for every neighbourhood U of (s_0,x_0). Let $U \subset \mathbb{R} \times X$ be a neighbourhood of (s_0,x_0). Then

$$(s_0 - \delta_1, s_0 + \delta_1) \times \{x \in X :\ |x(\omega_n) - x_0(\omega_n)| < \delta_2 \quad \text{for each } n \in \{1,\dots,N\}\} \subset U$$

with some $\delta_1, \delta_2 \in (0,\infty)$, $N \in \mathbb{N}$ and $\omega_1,\dots,\omega_N \in \mathbb{R}$. Choosing an

$$s \in (s_0 - \delta_1, s_0 + \delta_1) \setminus \{\omega_1,\dots,\omega_N\}$$

and then an $x \in X$ such that

$$|x(\omega_n) - x_0(\omega_n)| < \delta_2 \quad \text{for } n \in \{1,\dots,N\} \qquad \text{and} \qquad |x(s) - x_0(s_0)| \geq 1$$

we see that (s,x) is in U and

$$|F(s,x)(0) - F(s_0,x_0)(0)| = |x(s) - x_0(s_0)| \geq 1.$$

Hence $F(s,x)$ is not in W.

We conclude by posing the following open problem.

Problem. Do Theorem 1 and Corollary 1 remain valid without the separability assumptions on Y and X, respectively?

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References

[1] K. Baron and W. Jarczyk, *On continuity of solutions to some equations of iteration theory*, Aequationes Math. *69* (2005), 28–31.

[2] A. Járai, *Regularity properties of functional equations in several variables*, Adv. Math. *8*, Springer, New York, 2005.

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